RECITATION 3

INTRODUCING LIMITS

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Section 1. *Limits*

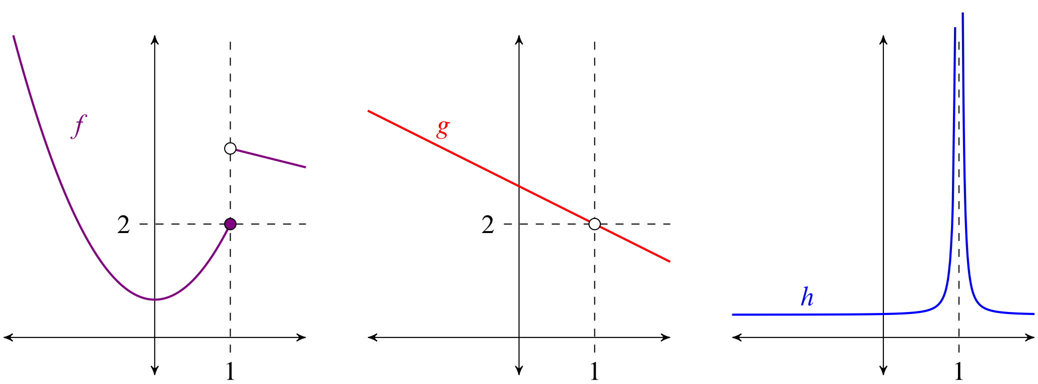
Informally, the limit of a function is where the function approaches, which is not necessarily where it ends up. The case where $\lim_{x \to a} f(x) = f(a)$ is extremely nice, and in this case we call f *continuous* (at a).

1 • 1. Result

For a function f, continuity is equivalent to the graph of f being connected, meaning you can in essence draw the graph of f without picking up your pencil.

In the case that f is continuous, everything is nice, and we can just evaluate limits $\lim_{x \to a} f(x)$ by evaluating f at a, the value x is going towards. The hard work is with discontinuous functions.

There are three things to worry about. This is best illustrated with the following graphs.



- f isn't continuous, because you can't connect $f(2)$ to the next piece.
- g isn't continuous, because there's a hole at $(1, 2)$, acting like a road block.
- h isn't continuous, because you can't connect the piece to the left of 1 to the piece on the right: ∞ isn't a number.

The issue with f is that if you approach $x = 1$ from the left, you get 2, while if you approach from the right, you get something else bigger than 2. In this case, there is no limit, because $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$.

The issue with g is that g isn't defined at 1: 1 isn't in the domain of g. Although it's clear that we can continue the line and define $g(1) = 2$, g itself isn't defined there. For our purposes in this class, g will often look something like

$$g(x) = \frac{(x-1)(\text{something}(x))}{(\text{other stuff}(x))(x-1)}.$$

So it's a matter of being able to factor the top and bottom, and cancel out any common factors ($(x-1)$ in this case). The resulting graph should remove the hole, and so be continuous, meaning that we can evaluate the limit just by plugging in $x = 1$.

The issue with h isn't that the limit doesn't exist: both sides approach the same thing. Nor is it that $h(1)$ "could" have been defined. Instead, the issue is that $\lim_{x\to 1} h(x)$ isn't a number, but instead infinity. Most of the time in this class, we can identify these because they will be of the form

$$h(x) = \frac{1}{\text{something}(x)} + \text{other stuff}(x).$$

Seeing whether it goes to infinity is then the same as seeing whether the something(x) goes to 0 and is positive, or whether the something(x) goes to 0 and is negative (if one side is positive while the other is negative, then you're like f, and the limit doesn't exist).

Section 2. Simple Methods for Evaluating Limits

There are basically four things to know for evaluating limits, beyond being able to do algebraic manipulations:

- $\lim_{x\to 0} \frac{\sin x}{x} = 1$;
- if f is continous at a, $\lim_{x\to a} f(x) = f(a)$;
- introduce (and remove) the conjugate if you see a square root: $\frac{\sqrt{x^2+a^2}-a}{\text{something}} = \frac{x^2}{\text{something}\cdot(\sqrt{x^2+a^2}+a)}$, by multiplying and dividing by the conjugate $\sqrt{x^2+a^2}+a$;
- cancel common factors if any appear: $\lim_{x\to a} \frac{(x-a)\cdot\text{something}}{(x-a)\cdot\text{other stuff}} = \lim_{x\to a} \frac{\text{something}}{\text{other stuff}}$.

Using these ideas along with trigonometric identities and the following will allow you to evaluate just about any limit given in the course at this point:

- $\lim_{x\to a}(f(x)\cdot g(x)) = (\lim_{x\to a} f(x))\cdot(\lim_{x\to a} g(x))$ if both limits are numbers.
- $\lim_{x\to a}(f(x)+g(x)) = (\lim_{x\to a} f(x)) + (\lim_{x\to a} g(x))$ if both limits are numbers.
- $\lim_{x\to a}(f(x)/g(x)) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if both limits are numbers (and we can divide by $\lim_{x\to a} g(x)$, i.e. if this limit of g isn't 0).

Note that the restrictions on these limits are necessary:

$$\lim_{x\to\infty}\left(x\cdot\frac{1}{x}\right) = \lim_{x\to\infty} 1 = 1$$

but $\lim_{x\to\infty} x = \infty$ while $\lim_{x\to\infty}\frac{1}{x} = 0$: $\infty\cdot 0$ is undefined since ∞ isn't a number. Indeed, taking $\frac{3}{x}$ instead of $\frac{1}{x}$ yields a limit of 3 instead of 1. So $\infty\cdot 0$ can't consistently be defined. Similarly,

$$\lim_{x\to\infty}(x+(-x)) = \lim_{x\to\infty} 0 = 0$$

but $\lim_{x\to\infty} x = \infty$ and $\lim_{x\to\infty}(-x) = -\infty$. Note that $\infty + (-\infty)$ isn't defined, since ∞ isn't a number. We could also have

$$\lim_{x\to\infty}(2x+(-x)) = \lim_{x\to\infty} x = \infty$$

although $\lim_{x\to\infty} 3x = \infty$ and $\lim_{x\to\infty}(-x) = -\infty$. If this all seems confusing, just remember that we need things to be numbers, and then everything works out properly.

Section 3. *Exercises*

Exercise 1

Evaluate the limit $\lim_{x \to -1} \dfrac{x^2 - 1}{x + 1}$.

Solution .:.

Note that $x^2 - 1 = (x+1)(x-1)$ and so $\frac{(x+1)(x-1)}{x+1} = x - 1$ for $x \neq -1$. Hence the equality holds everywhere else near $x = -1$, and so

$$\lim_{x \to -1} \frac{x^2 - 1}{x + 1} = \lim_{x \to -1} x - 1 = -2.$$

Exercise 2

Evaluate the limit $\lim_{x \to 0} \dfrac{\sin^2 x \cos^2 x}{x^2}$

Solution .:.

Note that $\lim_{x \to 0} \cos^2 x = 1$ by continuity. Similarly, $\lim_{x \to 0} \sin x / x = 1$ so that

$$\lim_{x \to 0} \frac{\sin^2 x \cos^2 x}{x^2} = \left(\lim_{x \to 0} \frac{\sin x}{x} \right) \left(\lim_{x \to 0} \frac{\sin x}{x} \right) \left(\lim_{x \to 0} \cos^2 x \right) = 1 \cdot 1 \cdot 1 = 1.$$

Exercise 3

Evaluate $\lim_{x \to 2} (\log_5(x^2 + x - 6) - \log_5(x - 2))$.

Solution .:.

Firstly, note that $x^2 + x - 6 = (x - 2)(x + 3)$. Hence for appropriate x so that everything is in the domain of log (you can check that this just requires $x > -3$),

$$\log_5(x^2 + x - 6) - \log_5(x - 2) = \log_5 \left(\frac{x^2 + x - 6}{x - 2} \right) = \log_5(x + 3).$$

So evaluating the limit yields $\lim_{x \to 2} \log_5(x + 3) = \log_5(5) = 1$.

Exercise 4

Evaluate $\lim_{x \to 0^+} \dfrac{\sin x}{\sqrt{x}}$.

Solution .:.

$\lim_{x \to 0^+} \sin x / x = 1$, as we know. Moreover, $\lim_{x \to 0^+} \sqrt{x} = 0$ and $\sqrt{x}^2 = |x| = x$ when $x > 0$. Hence

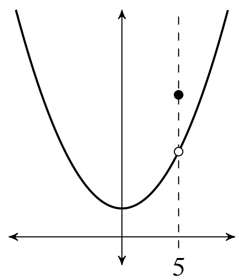
$$\lim_{x \to 0^+} \frac{\sin x}{\sqrt{x}} = \lim_{x \to 0^+} \frac{\sin(x)\sqrt{x}}{x} = \left(\lim_{x \to 0^+} \frac{\sin x}{x} \right) \left(\lim_{x \to 0^+} \sqrt{x} \right) = 1 \cdot 0 = 0.$$

Exercise 5

Draw the graph of a function f such that $f(5) \neq \lim_{x\to 5} f(x)$, and the limit exists.

Solution .:.

There are infinitely many such functions. Below is one such example:

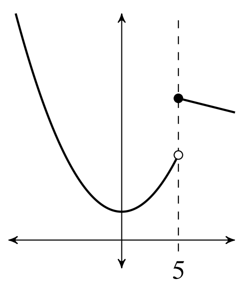


Exercise 6

Draw the graph of a function f such that $f(5) = \lim_{x\to 5^+} f(x)$ but $\lim_{x\to 5} f(x)$ does not exist.

Solution .:.

Again, just consider (a variant of) f from Section 1:



Exercise 7

Draw the graph of a function f such that $\lim_{x\to 0} f(x) = f(0)$.

Solution .:.

This is the easiest to do, as we just need to draw a function without picking up our pen (that is defined at 0):